

GENERIC SUBSTITUTIONS

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ABSTRACT. Up to equivalence, a substitution in propositional logic is an endomorphism of its free algebra. On the dual space, this results in a continuous function, and whenever the space carries a natural measure one may ask about the stochastic properties of the action. In classical logic there is a strong dichotomy: while over finitely many propositional variables everything is trivial, the study of the continuous transformations of the Cantor space is the subject of an extensive literature, and is far from being a completed task. In many-valued logic this dichotomy disappears: already in the finite-variable case many interesting phenomena occur, and the present paper aims at displaying some of these.

1. PRELIMINARIES

We work in the context of propositional algebraic logic. A *continuous t-norm* (t-norm for short) is a continuous function \star from $[0, 1]^2$ to $[0, 1]$ such that $([0, 1], \star, 1)$ is a commutative monoid for which $x \leq y$ implies $z \star x \leq z \star y$. Each t-norm induces a *residuum* \rightarrow via $x \leq (y \rightarrow z)$ iff $x \star y \leq z$. One checks easily that the usual lattice operations on $[0, 1]$ are term-definable by $x \wedge y = x \star (x \rightarrow y)$ and $x \vee y = ((x \rightarrow y) \rightarrow y) \wedge ((y \rightarrow x) \rightarrow x)$; therefore the determination of a continuous t-norm induces a uniquely defined structure of residuated lattice on $[0, 1]$.

Let L be a propositional language whose set of symbols for connectives contains $\star, \rightarrow, 1$. Let M be a structure for L satisfying:

- (A1) the support of M is a Borel subset of $[0, 1]$ which is either finite or of Lebesgue measure 1;
- (A2) the interpretations of \star and \rightarrow are the restrictions to M of a continuous t-norm and its residuum;
- (A3) the constant 1 is interpreted in the number $1 \in M$, while the other connectives —if any— are interpreted as functions on M of the appropriate arity that are Borel with respect to the topology that M inherits from $[0, 1]$;
- (A4) if M is finite, then $0 \in M$ and L contains a constant for 0.

Note that the residuum is Borel, since for every $0 < a < 1$ the set $\{(x, y) \in M^2 : x \rightarrow y \in [0, a]\}$ is open, while $\{(x, y) \in M^2 : x \rightarrow y \in (a, 1]\}$ is F_σ .

We are interested in the equational logic determined by M , i.e., the set $\text{EqTh}(M)$ of all term-identities $t(\bar{x}) = s(\bar{x})$ that hold in M . Let $\mathbf{V}(M)$ be the variety generated by M , i.e., the set of all algebras for L that satisfy $\text{EqTh}(M)$. Observe that M satisfies the quasi-identity

$$x \rightarrow y = 1 \ \& \ y \rightarrow x = 1 \implies x = y$$

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(indeed, assuming the premise, we have $x = x \star (x \rightarrow y) = y \star (y \rightarrow x) = y$). Therefore, by [24, Corollary 1.9], $\mathbf{V}(M)$ is ideal-determined, and the equational logic of M is just the algebraic counterpart of its assertional calculus [9]. In the following we list a few standard examples.

- Example 1.1.** (1) M is a chain $0 = a_0 < a_1 < \dots < a_{n-1} = 1$, the t-norm \star is interpreted as min, and a constant for 0 is added to the language. We obtain the n -valued Gödel-Dummet logic [20], which is boolean logic if $n = 2$;
- (2) \star and 0 as in (1), $M = [0, 1]$; we obtain the intuitionistic propositional calculus plus the prelinearity axiom $(x \rightarrow y) \vee (y \rightarrow x) = 1$;
- (3) M and \star as in (1), with a connective \sim added to the language and interpreted by $\sim a_i = a_{i+1} \pmod{n}$. We get Post's n -valued logic [49], [45, §2.3];
- (4) $M = (0, 1]$, \star = the ordinary product of real numbers. Then the residuum gets interpreted as $a \rightarrow b = \min\{1, b/a\}$. This is similar to the Product Logic in [25, §4.1], but since $0 \notin M$ every term induces a continuous function;
- (5) $M = [0, 1]$, \star the Łukasiewicz conjunction $a \star b = \max(0, a + b - 1)$: this is the logic of Wajsberg hoops [1];
- (6) as in (5), but a constant for 0 is added to the language; we obtain the Łukasiewicz infinite-valued logic;
- (7) $M = [0, 1]$. Denote temporarily the Łukasiewicz conjunction in (5) by \odot , and define a t-norm \star on M as follows:

$$a \star b = \begin{cases} 2^{-1}(2a \odot 2b), & \text{if } a, b < 1/2; \\ 2^{-1}(1 + ((2a - 1) \odot (2b - 1))), & \text{if } 1/2 \leq a, b; \\ \min\{a, b\}, & \text{otherwise.} \end{cases}$$

The resulting logic coincides with Hajek's Basic Logic [25, §2.2] on the set of all 1-variable formulas [35].

In all this paper, $\mathbf{V} = \mathbf{V}(M)$ will denote a variety of the above form. Every algebra $A \in \mathbf{V}$ carries a natural order given by $a \leq b$ iff $a \rightarrow b = 1$. A *filter* in A is the counterimage of 1 under some homomorphism of domain A . Ideal-determinacy of \mathbf{V} means that the lattice of congruences and the lattice of filters of A are isomorphic via $\theta \mapsto 1/\theta$. We say that a filter \mathfrak{p} of A is *prime* if, for any two filters $\mathfrak{f}, \mathfrak{g}$, if $\mathfrak{p} \supseteq \mathfrak{f} \cap \mathfrak{g}$ then $\mathfrak{p} \supseteq \mathfrak{f}$ or $\mathfrak{p} \supseteq \mathfrak{g}$. Since a lattice structure is term-interpretable in all algebras of \mathbf{V} , the variety \mathbf{V} is congruence-distributive [10, Theorem 12.3], and \mathfrak{p} is prime iff the implication

$$\mathfrak{p} = \mathfrak{f} \cap \mathfrak{g} \implies \mathfrak{p} = \mathfrak{f} \text{ or } \mathfrak{p} = \mathfrak{g}$$

holds.

A filter $\mathfrak{f} = (a)$ is *principal* if it is generated by a single element a ; in other words, (a) is the intersection of all filters that contain a . An *ideal term in y* is a term $t(x_1, \dots, x_n, y)$ such that the equation $t(x_1, \dots, x_n, 1) = 1$ is valid in \mathbf{V} . By [24, Lemma 1.2],

$$(a) = \{t(c_1, \dots, c_n, a) : t(\bar{x}, y) \text{ is an ideal term in } y \text{ and } c_1, \dots, c_n \in A\}.$$

Proposition 1.2. $(a) \cap (b) = (a \vee b)$.

Proof. The right-to-left inclusion is clear. Let $t(\bar{x}, y), s(\bar{z}, w)$ be terms, not necessarily ideal. Then the equation

$$t(\bar{x}, y \vee w) \wedge s(\bar{z}, y \vee w) \leq t(\bar{x}, y) \vee s(\bar{z}, w) \quad (*)$$

is true in M . Indeed, if y, w are interpreted in l, m , then $y \vee w$ is interpreted either in l or in m . Since \mathbf{V} is generated by M , $(*)$ holds in \mathbf{V} . Assume now that $e = t(\bar{c}, a) = s(\bar{d}, b) \in (a) \cap (b)$, for certain ideal terms $t(\bar{x}, y), s(\bar{z}, w)$ in y and w , respectively, and certain $\bar{c}, \bar{d} \in A$. Then the term $r(\bar{x}, \bar{z}, u) = t(\bar{x}, u) \wedge s(\bar{z}, u)$ is ideal in u , and we have

$$e = t(\bar{c}, a) \vee s(\bar{d}, b) \geq r(\bar{c}, \bar{d}, a \vee b) \in (a \vee b).$$

Since clearly filters are closed upwards, $e \in (a \vee b)$ and the proof is complete. \square

Proposition 1.3. *Let $A \in \mathbf{V}$. The following are equivalent:*

- (i) *A is totally-ordered;*
- (ii) *the set of filters of A is totally-ordered;*
- (iii) *every filter is prime;*
- (iv) *the trivial filter (1) is prime.*

The above conditions are implied by —but do not imply— subdirect irreducibility of A .

Proof. The implications (i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) are clear. Assume (iv), and let $a, b \in A$. We have $1 = (a \rightarrow b) \vee (b \rightarrow a)$, and therefore by Proposition 1.2 (1) = $(a \rightarrow b) \cap (b \rightarrow a)$. Since the filter (1) is prime, it must be either $a \leq b$ or $b \leq a$, and therefore A is totally-ordered. If A is subdirectly irreducible, then trivially the filter (1) is prime. Let $A = ([0, 1], \star, \rightarrow, 0, 1)$ be as in Example 1.1(2), θ a nontrivial congruence on A , and let $\mathfrak{f} = 1/\theta \neq (1)$ be the filter associated to θ . Then there exists a unique $a \in [0, 1)$ such that \mathfrak{f} equals either $(a, 1]$ or $[a, 1]$, and $\theta = \text{diag}(A) \cup \mathfrak{f}^2$. Indeed, assume $b\theta c$ and, without loss of generality, $b \leq c$. Then $b = c \rightarrow b \in \mathfrak{f}$ and therefore both b and c are in \mathfrak{f} . By defining $a = \inf \mathfrak{f}$, our description of \mathfrak{f} and θ follows. It is immediate that A is a totally-ordered subdirectly reducible algebra. \square

Let $\text{Spec } A$ be the set of all proper prime filters of A , excluding the improper filter A . Endow $\text{Spec } A$ with the hull-kernel topology, by taking the family of all sets of the form

$$O_a = \{\mathfrak{p} \in \text{Spec } A : a \notin \mathfrak{p}\}$$

as an open subbasis. The following properties are well known [28], [5, Chapter 10], [50]:

- (1) $\text{Spec } A$ is a —possibly noncompact— *spectral space*, i.e., a space possessing an intersection-closed basis of compact open sets, and in which every irreducible closed set is the closure of a unique point;
- (2) the closure of $Y \subseteq \text{Spec } A$ is $\{\mathfrak{p} \in \text{Spec } A : \mathfrak{p} \supseteq \bigcap Y\}$, and in particular Y is dense iff $\bigcap Y = \{(1)\}$. The relation $\mathfrak{p} \leq \mathfrak{q}$ iff $\mathfrak{p} \subseteq \mathfrak{q}$ iff \mathfrak{q} is in the closure of \mathfrak{p} is the *specialization order* of $\text{Spec } A$;
- (3) every filter of A is a —possibly empty— intersection of proper prime filters (because the proper filter \mathfrak{p} is prime if A/\mathfrak{p} is subdirectly irreducible). Hence the mapping

$$\mathfrak{f} \mapsto O_{\mathfrak{f}} = \{\mathfrak{p} \in \text{Spec } A : \mathfrak{f} \not\subseteq \mathfrak{p}\}$$

is an isomorphism between the lattice of filters of A and the lattice of open sets in $\text{Spec } A$;

- (4) under the above mapping, the finitely generated filters correspond to the compact open sets. By Proposition 1.2, these are closed under finite intersections. In particular, $\text{Spec } A$ is compact iff A is finitely generated as a filter (an example of an algebra with a noncompact spectrum is the one used in the proof of Proposition 1.3, as soon as one removes the constant 0 from the language).

2. SUBSTITUTIONS ACTING ON SPECTRA

A *syntactical substitution* is an endomorphism of the algebra of terms in L . Since we work up to the equational theory of \mathbf{V} , we are actually interested in endomorphisms of the free algebra in \mathbf{V} over $\kappa \leq \omega$ many generators; we denote this latter algebra by $\text{Free}_\kappa(\mathbf{V})$. Endomorphisms of $\text{Free}_\kappa(\mathbf{V})$ will be named *substitutions over κ variables*; a substitution is *invertible* if it is an automorphism of $\text{Free}_\kappa(\mathbf{V})$. It is well known that $\text{Free}_\kappa(\mathbf{V})$ is the subalgebra of M^{M^κ} generated by the projection functions $x_i : M^\kappa \rightarrow M$, for $0 \leq i < \kappa$. We give to M the topology induced by the standard topology on $[0, 1]$, and to M^κ the product topology.

Let λ be the probability measure naturally associated with M . More precisely, if M has Lebesgue measure 1 as a subset of $[0, 1]$, then λ is the restriction of the Lebesgue measure to M , while if M is finite then λ is the counting measure; in both cases $\lambda(M) = 1$. We extend λ to a probability measure on M^κ by defining

$$\lambda(C(A_{i_1}, \dots, A_{i_t})) = \lambda(A_{i_1}) \times \dots \times \lambda(A_{i_t}),$$

where the A_{i_j} s are measurable subsets of M , and $C(A_{i_1}, \dots, A_{i_t})$ is the cylinder $\{p \in M^\kappa : x_{i_j}(p) \in A_{i_j} \text{ for every } j = 1, \dots, t\}$.

For every $p \in M^\kappa$, we have the homomorphism $\text{Free}_\kappa(\mathbf{V}) \rightarrow M$ given by evaluation at p . Let $\pi(p)$ be the kernel, which is a prime filter by Proposition 1.3, since M is totally-ordered. For every $t \in \text{Free}_\kappa(\mathbf{V})$, let $Z(t) = \{p \in M^\kappa : t(p) = 1\}$ be the 1-set of t . The mapping π is Borel: indeed, the counterimage of the basic closed set $F_t = \text{Spec } \text{Free}_\kappa(\mathbf{V}) \setminus O_t$ is $Z(t)$, which is Borel by our assumption (A3). Note that π is undefined at the points $p \in \bigcap \{Z(t) : t \in \text{Free}_\kappa(\mathbf{V})\} = Z_0$, since then $\pi(p)$ is the improper filter. This gives no trouble, since Z_0 is either empty (e.g., if L contains a constant for 0), or contains the element $p_0 = (\dots, 1, \dots)$ only, and in this latter case $\lambda(Z_0) = 0$ by our assumption (A4).

We push forward λ to a Borel probability measure on $\text{Spec } \text{Free}_\kappa(\mathbf{V})$ by setting

$$\lambda(A) = \lambda(\pi^{-1}[A]).$$

We use λ both for the original measure on M^κ and for the induced measure on $\text{Spec } \text{Free}_\kappa(\mathbf{V})$; this should cause no confusion.

We briefly recall a few basic facts of ergodic theory [53], [15], [48]: we state them at a level of generality appropriate to our setting. A *measure-theoretic dynamical system* is a triple (X, μ, S) where X is a second countable topological space, μ is a Borel probability measure on X (i.e., $\mu(X) = 1$ and μ is defined on the Borel σ -algebra \mathfrak{B}), and $S : X \rightarrow X$ is a measurable mapping. We let A, B, \dots vary over the elements of \mathfrak{B} , and we drop universal quantifications over these objects in our definitions. We say that S is *nonsingular* if $\mu(A) = 0$ implies $\mu(S^{-1}[A]) = 0$, and *measure-preserving* if $\mu(A) = \mu(S^{-1}[A])$. Assume that S is measure-preserving. Then S is:

- (1) *ergodic* if $S^{-1}[A] = A$ implies $\mu(A) = 0$ or $\mu(A) = 1$;
- (2) *mixing* if

$$\lim_{n \rightarrow \infty} \mu(A \cap S^{-n}[B]) = \mu(A) \cdot \mu(B);$$

- (3) *exact* if the tail σ -field $\bigcap_{n \geq 0} S^{-n}\mathfrak{B}$ contains only sets of measure 0 or 1.

It is a classical fact that exactness \Rightarrow mixing \Rightarrow ergodicity. We single out in the following proposition the property of ergodic systems that mainly concerns us.

Proposition 2.1. *Let (X, μ, S) be an ergodic system, and assume that μ is supported on all of X (i.e., every nonempty open set has measure > 0). Then for μ -all points $x \in X$ the following holds:*

- (*) *for every $k \geq 0$, the point $S^k(x)$ has a dense orbit.*

Proof. One of several characterizations of ergodicity is that $\mu(A) > 0$ implies $\mu(\bigcup\{S^{-n}[A] : 0 \leq n\}) = 1$ [53, Theorem 1.5]. Since S is measure-preserving, for all $k \geq 0$ the set $B(A, k) = \bigcup\{S^{-n}[A] : k \leq n\}$ has also measure 1. Let $\{A_i : i < \omega\}$ be a countable basis of nonempty open sets. Then the set of all x that satisfy property (*) is $\bigcap\{B(A_i, k) : i, k < \omega\}$, which has measure 1. \square

Let $\sigma : \text{Free}_\kappa(\mathbf{V}) \rightarrow \text{Free}_\kappa(\mathbf{V})$ be a substitution. We always assume that σ is *nontrivial*, i.e., that the counterimage of every proper prime filter is proper; this condition is automatically satisfied if 0 is a constant of the language. Let X be $\text{Spec Free}_\kappa(\mathbf{V})$, λ the probability measure on X defined at the beginning of this section, S the transformation given by $S(\mathbf{p}) = \sigma^{-1}[\mathbf{p}]$. By our assumption $\kappa \leq \omega$, the space X is second countable. Since $S^{-1}[F_t] = F_{\sigma(t)}$, we have that S is continuous and (X, λ, S) is a measure-theoretic dynamical system. Let us say that σ is *minimal* if in the associated dynamical system (X, S) all points $x \in X$ have a dense orbit [53, Definition 5.1]. The minimality property (which is purely topological, since no measure is involved) is quite relevant to logic. Indeed, let $t(\bar{x}) = s(\bar{x})$ be an n -variable identity in the language L . Then $t(\bar{x}) = s(\bar{x})$ is true in \mathbf{V} iff it is true in the generators of $\text{Free}_n(\mathbf{V})$ iff t and s are equal modulo \mathbf{p} , for \mathbf{p} ranging on a dense subset of $\text{Spec Free}_n(\mathbf{V})$. Let σ be a minimal substitution over n variables. Then in order to test the truth of $t(\bar{x}) = s(\bar{x})$ we may choose any prime filter \mathbf{p} and test the identity modulo all points of the S -orbit of \mathbf{p} (dually stated, we test all identities $\sigma^n(t) = \sigma^n(s)$, for $n \geq 0$, modulo a fixed \mathbf{p}). For all \mathbf{p} , this procedure provides a correct test.

Most many-valued logics do not admit minimal substitutions. For example, we shall see that both for Łukasiewicz and for product logic the space of maximal filters is homeomorphic to a closed Euclidean ball on which S acts continuously; such an action necessarily has fixed points by the Brouwer fixed point theorem. We are therefore lead to taking into consideration measure-theoretic issues.

Definition 2.2. Let $X = \text{Spec Free}_\kappa(\mathbf{V})$, λ be as above. We say that a Borel probability measure μ on X is *algebraically equivalent* to λ if there exists an automorphism ρ of $\text{Free}_\kappa(\mathbf{V})$ such that, writing R for the dual homeomorphism, we have $\mu(A) = \lambda(R^{-1}[A])$ for every A . Let σ be a substitution, S its dual. We say that σ is *generic* if for every μ algebraically equivalent to λ the set $\{x \in X : x \text{ has a dense } S\text{-orbit}\}$ has μ -measure 1.

Remark 2.3. The reader should compare the notion of algebraic equivalence of measures with that of topological equivalence in [42, § 3]. The two notions coincide if all homeomorphisms of X come from automorphisms of $\text{Free}_\kappa(\mathbf{V})$: this is

the case, e.g., of classical logic. For the two cases that mainly concern us, namely Łukasiewicz logic and falsum-free product logic over finitely many variables, algebraic equivalence gives essentially no new measures. Indeed, by [18, Theorem 2.6] and our Theorem 3.1, λ is invariant under the action of the automorphism group of Łukasiewicz logic. Analogously, we will see in Section 4 that formulas of falsum-free product logic correspond to piecewise-linear homogeneous functions over the negative cone of \mathbb{R}^n . Automorphisms of such structures are induced by finite families of nonsingular $n \times n$ matrices, as explained in [4]. This fact and Theorem 4.3 imply that all measures algebraically equivalent to λ share the same σ -ideal of nullsets. I thank Andrew Glass and Daniele Mundici for clarifying discussions over this point.

We think of genericity as a replacement for minimality, for logics that do not admit minimal substitutions. When testing an identity along the orbit of a prime filter, the initial choice of the filter becomes relevant; however, from the measure-theoretic point of view, all choices are good. Note that in certain pathological situations the condition (*) of Proposition 2.1 results to be stronger than genericity, since it neglects the transient behaviour of the orbits; see Example 2.4.

Let $s_i = \sigma(x_i)$. Then the κ -tuple (\dots, s_i, \dots) determines a function $\bar{s} : M^\kappa \rightarrow M^\kappa$ via $p \mapsto (\dots, s_i(p), \dots)$, and the following diagram commutes

$$\begin{array}{ccc} M^\kappa & \xrightarrow{\bar{s}} & M^\kappa \\ \pi \downarrow & & \downarrow \pi \\ X & \xrightarrow{S} & X \end{array}$$

Note that the diagram is valid even when π is undefined at $p_0 = (\dots, 1, \dots)$. Indeed, since σ is nontrivial we have $\bar{s}^{-1}[\{p_0\}] = \{p_0\}$, and we may safely substitute M^κ in the top row with $M^\kappa \setminus \{p_0\}$. The mapping S is continuous, while \bar{s} and π are in general only Borel. Each of the maps σ and \bar{s} determines the other; moreover, \bar{s} determines S since π has dense range. If π is injective then S determines \bar{s} . This may fail if π not injective: one easily constructs an example by using the logic of Example 1.1(4). Its 1-generated free algebra is the chain $1 > x > x \star x > x \star x \star x > \dots$, whose spectrum is the single point (1) . The substitution $x \mapsto 1$ is trivial, while every other substitution induces the identity map on the spectrum. Note however that π is injective for many relevant cases, e.g., for classical logic (where it is even a homeomorphism) and for Łukasiewicz logic. Let us present here a pathological example.

Example 2.4. Let M be as in Example 1.1(2), $\mathbf{V} = \mathbf{V}(M)$. Then $\text{Free}_1(\mathbf{V})$ is isomorphic to the direct product $\{0, 1\} \times \{0, 1/2, 1\}$ (the factors being subalgebras of M), with free generator $(0, 1/2)$. This is easily seen by observing that:

- (1) for every $0 < a < 1$, the subalgebra of M generated by a is isomorphic to $\{0, 1/2, 1\}$;
- (2) for every term $t(x)$ we have $t(1) = 1$ iff $t(1/2) \geq 1/2$.

Hence the subalgebra of $[0, 1]^{[0, 1]}$ generated by the identity function is isomorphic to $\{0, 1\} \times \{0, 1/2, 1\} \times \{0, 1\}$ (by (1)), but the third factor is superfluous (by (2)). By the way, an analogous argument shows that $\text{Free}_n(\mathbf{V})$ is a finite product of finite chains; this fact was proved in [29]. Under the specialization order, $X = \text{Spec } \text{Free}_1(\mathbf{V})$ is the poset



where

- $\mathbf{m}_0 = \{1\} \times \{0, 1/2, 1\} = \pi(0)$;
- $\mathbf{p} = \{0, 1\} \times \{1\} = \pi(a)$, for every $a \in (0, 1)$;
- $\mathbf{m}_1 = \{0, 1\} \times \{1/2, 1\} = \pi(1)$.

Note that λ is concentrated in \mathbf{p} : $\lambda(A) = 1$ iff $\mathbf{p} \in A$. Let σ be the substitution that maps x to $s = x \rightarrow 0$. Then $\bar{s}(0) = 1$ and $\bar{s}(a) = 0$ for $a \in (0, 1]$; also $S(\mathbf{p}) = S(\mathbf{m}_1) = \mathbf{m}_0$ and $S(\mathbf{m}_0) = \mathbf{m}_1$. Both \bar{s} and S are singular. The S -orbit of \mathbf{p} is all of X , and hence σ is generic. On the other hand, the condition $(*)$ of Proposition 2.1 is not satisfied, because the transient behaviour is relevant: the orbit of $S(\mathbf{p})$ is not dense.

Recall that a measure ν is *absolutely continuous* w.r.t. a measure μ on the same space if $\mu(A) = 0$ implies $\nu(A) = 0$; we write then $\nu \ll \mu$.

Definition 2.5. Let σ be a substitution, S its associated continuous mapping on $X = \text{Spec Free}_\kappa(\mathbf{V})$. We say that σ is *ergodic* (respectively, *mixing* or *exact*) if there exists a Borel probability measure μ on X such that:

- (1) all measures algebraically equivalent to λ are $\ll \mu$;
- (2) (X, μ, S) is ergodic (respectively, mixing or exact).

Remark 2.6. (1) Every ergodic substitution is generic, but the converse is false: see Theorem 4.4(3).

- (2) The introduction of the auxiliary measure μ is unavoidable; in Example 3.2 we shall see two ergodic substitutions whose dual maps do not preserve λ .
- (3) The reader may wonder why we did not define σ to be generic if the set of points of M^κ that have a dense orbit under \bar{s} has measure 1 under all measures algebraically equivalent to λ . This definition is simpler, since it avoids the introduction of the machinery of spectra, and stronger, since a substitution which is generic according to it is also generic in our sense. The trouble is that it is too strong: the mapping π may collapse many points, so that \bar{s} does not act generically on the points of M^κ , although it does on the fibers of π . Stated otherwise: the orbit of a random point of M^κ may be not dense in M^κ , but still contain sufficiently many points to ascertain logical truth. We shall see an example in Theorem 4.4(3.2).

In the remainder of this section, we discuss the familiar case of classical logic. With the obvious modifications, our discussion applies to Post's n -valued logic as well, because both logics are functionally complete. Let then \mathbf{V} be the variety of boolean algebras. The mapping π is a homeomorphism that identifies the discrete set 2^n and the Cantor space 2^ω with $\text{Spec Free}_n(\mathbf{V})$ and $\text{Spec Free}_\omega(\mathbf{V})$, respectively. In the first case λ is the counting measure (which is obviously fixed by algebraic equivalence), and in the second is the measure determined by

$$\lambda(C(e_{i_1}, \dots, e_{i_t})) = 2^{-t},$$

where $i_1, \dots, i_t < \omega$, $e_{i_1}, \dots, e_{i_t} \in \{0, 1\}$, and $C(e_{i_1}, \dots, e_{i_t})$ is the clopen cylinder $\{p \in 2^\omega : p(i_j) = e_{i_j}\}$. Every substitution is nontrivial, and every continuous mapping from 2^κ to itself is induced by a unique substitution. For substitutions over finitely many variables the situation trivializes.

Proposition 2.7. *No substitution over $n < \omega$ variables is exact or mixing. For every such substitution σ , the following are equivalent:*

- (i) σ is ergodic;
- (ii) σ is generic;
- (iii) σ is minimal;
- (iv) S permutes cyclically the 2^n elements of $\text{Spec Free}_n(\mathbf{V})$;
- (v) σ is invertible and the group it generates has order 2^n .

Proof. If S preserves a measure $\mu \gg \lambda$, then it must be surjective. Hence it is bijective and $\mu = \lambda$; clearly S cannot be mixing or exact. (i) \Rightarrow (ii) always holds, and (ii) \Rightarrow (iii) holds because every point has nonzero measure. By following the orbit of a point one sees that a minimal S satisfies (iv). (iv) \Rightarrow (v) trivially, and (iv) \Rightarrow (i) because S is then ergodic with respect to λ . Assume (v), and decompose S in the product of disjoint cycles in the symmetric group over 2^n letters. The order of S is the least common multiple of the lengths of the cycles. Therefore each cycle has length a power of 2, which in turn implies that there is only one cycle. Hence (iv) holds. \square

As we noted in the Abstract, the study of continuous transformations of the Cantor space, i.e., in our language, of substitutions over ω variables in classical logic, is the subject of an extensive literature. The simplest minimal—hence generic—substitutions are those given by adding machines. Realize the Cantor space as the space underlying the topological group \mathbb{Z}_p of p -adic integers, for some prime p . Let $\alpha \in \mathbb{Z}_p$ be such that $\chi(\alpha) \neq 1$ for every nontrivial continuous character χ ; this amounts to the invertibility of α in the local ring \mathbb{Z}_p . Then [15, pp. 97–99] guarantees that the translation by α ($\beta \mapsto \alpha + \beta$) is a minimal homeomorphism of \mathbb{Z}_p . The simplest choice is to take $\alpha = 1 \in \mathbb{Z}_2$, and by explicit computation one sees that the corresponding substitution σ is given by

$$\begin{aligned} x_0 &\mapsto \neg x_0 \\ x_{i+1} &\mapsto ((x_0 \wedge x_1 \wedge \dots \wedge x_i) \leftrightarrow \neg x_{i+1}) \end{aligned}$$

Note that restricting σ to the first n variables gives a substitution of the form described in Proposition 2.7, namely the translation by 1 in \mathbb{Z}_{2^n} .

All minimal transformations of the Cantor space can be realized as mappings on the path spaces of certain combinatorial objects, called Bratteli-Vershik diagrams [27, Theorem 4.7]. Moreover, a unique dimension group with order unit [21] can be read off either from the transformation or from a corresponding diagram [27, Theorem 5.4 and Corollary 6.3]; this dimension group is a complete invariant for strong orbit equivalence of Cantor minimal systems [23]. It is remarkable that Bratteli diagrams and dimension groups play a rôle also in Łukasiewicz logic [37], [44, §4.2]. It is unclear whether this recurrent appearance is a coincidence or an indicator of some hidden underlying structure.

3. LUKASIEWICZ LOGIC

We have seen in the previous section that for classical and Post logics the generic substitutions are either trivial (in the finite-variable case) or extremely complex (in the infinite-variable case). In this section we shall describe completely the stochastic properties of 1-variable substitutions in Łukasiewicz logic. In this logic, the 1-variable case has always been an excellent test case [39], [41], for two main reasons:

- it presents the piecewise-linear structure of the free algebras in a simplified nontrivial fashion;
- the extensions of Łukasiewicz logic —i.e., the subvarieties of the variety \mathbf{MV} of MV-algebras— are axiomatizable by 1-variable identities [31], [19], [47], and hence correspond bijectively to the fully invariant filters of $\text{Free}_1(\mathbf{MV})$ [10, Corollary 14.10].

In this section $\mathbf{MV} = \mathbf{V}(M)$ will denote the variety of Example 1.1(6), whose elements are named MV-algebras. We presuppose familiarity with the basic theory of MV-algebras; see [12], [13] for a quick introduction or [11] for a more extensive treatment.

A *McNaughton function over the n -cube* [34], [40] is a continuous function $t : [0, 1]^n \rightarrow [0, 1]$ for which the following holds:

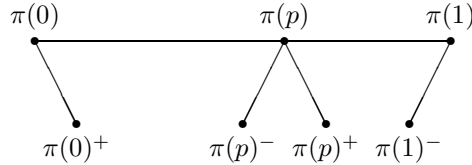
there exist finitely many affine linear polynomials t_1, \dots, t_k , each t_i of the form $t_i = a_i^0 x_0 + a_i^1 x_1 + \dots + a_i^{n-1} x_{n-1} + a_i^n$, with a_i^0, \dots, a_i^n integers, such that, for each $p \in [0, 1]^n$, there exists $i \in \{1, \dots, k\}$ with $t(p) = t_i(p)$.

It is well known that the free MV-algebra over n generators $\text{Free}_n(\mathbf{MV})$ is the algebra of all McNaughton functions over the n -cube under pointwise operations. Note that we tacitly identify an n -variable term with the function it induces.

Let σ be a substitution over n variables. As in Section 2 we have the commuting diagram

$$\begin{array}{ccc} [0, 1]^n & \xrightarrow{\bar{s}} & [0, 1]^n \\ \pi \downarrow & & \downarrow \pi \\ \text{Spec Free}_n(\mathbf{MV}) & \xrightarrow{s} & \text{Spec Free}_n(\mathbf{MV}) \end{array}$$

By [36, Proposition 8.1] π maps homeomorphically the n -cube to the subspace of maximal elements of $\text{Spec Free}_n(\mathbf{MV})$. As an example, we draw a picture of $\text{Spec Free}_1(\mathbf{MV})$ under the specialization order:



For every rational point $p \in (0, 1)$, there are two prime filters $\pi(p)^+$ and $\pi(p)^-$ attached below the maximal $\pi(p)$. If p is irrational, then $\pi(p)$ is a minimal prime filter. Finally, there are two points $\pi(0)^+$ and $\pi(1)^-$ attached below $\pi(0)$ and $\pi(1)$. The prime filter $\pi(p)^-$ is the filter of all McNaughton functions that are 1 in a left

neighborhood of p , and analogously for $\pi(p)^+$, $\pi(0)^+$, and $\pi(1)^-$. See [46] for a generalization of this description to $\text{Spec Free}_n(\mathbf{MV})$.

In the cases of classical and Post logics, we saw that π provides a topological conjugacy between \bar{s} and S . This is definitely false in Łukasiewicz logic, since the spaces $[0, 1]^n$ and $\text{Spec Free}_n(\mathbf{MV})$ are not even homeomorphic. Nevertheless, as we shall see in Theorem 3.1, Łukasiewicz logic enjoys the nice property that \bar{s} and S are measure-theoretically isomorphic (provided that \bar{s} is nonsingular, which is of course the interesting case). As a consequence, we can carry out our quest of stochastic properties at the simpler level of transformations of the n -cube.

Recall [53, Definition 2.4] that two measure-theoretic dynamical systems (X_i, μ_i, S_i) , for $i = 1, 2$, are *isomorphic* if there are Borel subsets $A_i \subseteq X_i$ of full measure such that

- (1) $S_i[A_i] \subseteq A_i$;
- (2) there exists a measure-preserving Borel isomorphism $\rho : A_1 \rightarrow A_2$ such that $\rho \circ S_1 = S_2 \circ \rho$.

Theorem 3.1. *Consider the systems $([0, 1]^n, \lambda, \bar{s})$ and $(\text{Spec Free}_n(\mathbf{MV}), \lambda, S)$ as above. Then \bar{s} is nonsingular iff S is nonsingular. If this happens, the two systems are isomorphic under π .*

Proof. If S is singular, then \bar{s} is singular by the definition of the measure λ on $\text{Spec Free}_n(\mathbf{MV})$. Consider now \bar{s} . By [34, p. 2] and [18, Theorem 2.6] there exists a finite partition $\{P_1, \dots, P_r\}$ of the n -cube in compact convex polyhedrons of dimension n such that \bar{s} is expressible by

$$\begin{pmatrix} \alpha_0 \\ \vdots \\ \alpha_{n-1} \end{pmatrix} \mapsto U_k \begin{pmatrix} \alpha_0 \\ \vdots \\ \alpha_{n-1} \end{pmatrix} + V_k$$

on each polyhedron P_k ; in the above expression U_k is an $n \times n$ matrix and V_k a column vector, both having integer entries. Suppose that \bar{s} is singular. Then U_k must be singular for some k , and hence $\bar{s}[P_k]$ is a polyhedron of dimension $< n$. Let $t \in \text{Free}_n(\mathbf{MV})$ be such that the 1-set $Z(t)$ of t is exactly $\bar{s}[P_k]$; such a t exists by the theory of Schauder hats [40], [43]. Let $F_t = \text{Spec Free}_n(\mathbf{MV}) \setminus O_t$ be the basic closed set determined by t . Then $\lambda(F_t) = \lambda(\pi^{-1}[F_t]) = \lambda(Z(t)) = 0$. On the other hand

$$\begin{aligned} \lambda(S^{-1}[F_t]) &= \lambda(F_{\sigma(t)}) \\ &= \lambda(Z(\sigma(t))) \\ &\geq \lambda(P_k) \\ &> 0, \end{aligned}$$

since $\sigma(t) = t \circ \bar{s}$ and $P_k \subseteq Z(t \circ \bar{s})$; hence S is singular.

We now assume that \bar{s} and S are nonsingular, and prove our second statement. For every $t \in \text{Free}_n(\mathbf{MV})$, let

$$D_t = \bigcap \{U : U \text{ is an open set in } \text{Spec Free}_n(\mathbf{MV}) \text{ and } U \supseteq F_t\}.$$

Claim. D_t is Borel (actually G_δ) and $\mathfrak{p} \in D_t$ iff the maximal filter to which \mathfrak{p} specializes (i.e., the unique maximal \mathfrak{m} such that $\mathfrak{m} \supseteq \mathfrak{p}$) is in F_t .

Proof of Claim. Let U be an open set containing F_t , and write $U = \bigcup \{O_{r_i} : i \in I\}$. Since F_t is closed and $\text{Spec Free}_n(\mathbf{MV})$ is compact (because $\text{Free}_n(\mathbf{MV})$ is finitely

generated as an improper filter), there exists a finite subset J of I such that, writing $r = \bigwedge_{i \in J} r_i$, we have $F_t \subseteq O_r \subseteq U$. This implies that D_t is a (necessarily countable) intersection of basic open sets. Assume now that the maximal specialization \mathfrak{m} of \mathfrak{p} is not in F_t , and let $\mathfrak{m} = \pi(p)$ for a uniquely determined $p \in [0, 1]^n$. Let E be a closed n -dimensional ball centered in p and having empty intersection with $Z(t)$. By [38, Corollary 3.4] there exists $q \in \text{Free}_n(\mathbf{MV})$ such that $q \upharpoonright E = 1$ and $Z(q) \cap Z(t) = \emptyset$. The latter identity implies $F_q \cap F_t = \emptyset$, and the former implies $q \in \mathfrak{p}$ by [38, Proposition 3.1]. Therefore $\mathfrak{p} \notin O_q \supseteq F_t$, and $\mathfrak{p} \notin D_t$. Since open sets are downwards closed in the specialization order, the reverse inclusion $\mathfrak{m} \in F_t \Rightarrow \mathfrak{p} \in D_t$ is immediate. \square

Let \mathcal{Z} be the set of all $t \in \text{Free}_n(\mathbf{MV})$ such that $\lambda(Z(t)) = 0$. Then, for every $t \in \mathcal{Z}$,

$$\begin{aligned} \lambda(D_t) &= \lambda(\pi^{-1}[\bigcap\{U : U \text{ is open and } U \supset F_t\}]) \\ &= \lambda(\bigcap\{\pi^{-1}[U] : U \text{ is open and } U \supset F_t\}) \\ &= \lambda(\bigcap\{V \subseteq [0, 1]^n : V \text{ is open and } V \supseteq Z(t)\}) \\ &= 0, \end{aligned}$$

since the Lebesgue measure on $[0, 1]^n$ is regular. Let

$$\begin{aligned} B &= \bigcup\{Z(t) : t \in \mathcal{Z}\} \\ C &= \bigcup\{D_t : t \in \mathcal{Z}\} \end{aligned}$$

Then C is the set of all $\mathfrak{p} \in \text{Spec Free}_n(\mathbf{MV})$ whose maximal specialization $\mathfrak{m} = \pi(p)$ is such that $p \in B$. In particular $\pi^{-1}[C] = B$. Let $A_1 = [0, 1]^n \setminus B$, $A_2 = \text{Spec Free}_n(\mathbf{MV}) \setminus C$. Then:

- $p \in A_1$ iff $\pi(p) \in A_2$;
- A_1 and A_2 are Borel sets of full measure;
- $\pi \upharpoonright A_1$ is surjective on A_2 . Indeed, let $\mathfrak{p} \in A_2$ and let $\pi(p)$ be its maximal specialization. Then $p \notin B$. Write $p = (\alpha_0, \dots, \alpha_{n-1}) \in [0, 1]^n$. The real numbers $\alpha_0, \dots, \alpha_{n-1}, 1$ must be linearly independent over \mathbb{Q} since, otherwise, we would be able to find $t \in \mathcal{Z}$ such that $t(p) = 1$. By [46, Theorem 4.8 and Corollary 4.9], no element of $\text{Spec Free}_n(\mathbf{MV})$ specializes to $\pi(p)$ except $\pi(p)$ itself. Hence $\mathfrak{p} = \pi(p)$;
- $\bar{s}[A_1] \subseteq A_1$ and $S[A_2] \subseteq A_2$. Indeed, we only need to show that $\bar{s}(p) \in B$ implies $p \in B$. Let $\bar{s}(p) \in Z(t)$, for some $t \in \mathcal{Z}$. Since $Z(t \circ \bar{s}) = \bar{s}^{-1}[Z(t)]$ and \bar{s} is nonsingular, $t \circ \bar{s} \in \mathcal{Z}$. Therefore $p \in Z(t \circ \bar{s}) \subseteq B$.

Since π is injective and measure-preserving, its restriction to A_1 is an isomorphism as required. \square

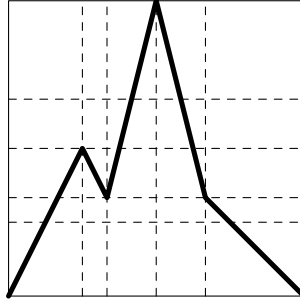
In our next theorem we will characterize the generic substitutions over 1 variable. Let $s \in \text{Free}_1(\mathbf{MV})$ and let $Q_0 \subset \mathbb{Q}$ be the finite set of points of nondifferentiability of s , including 0 and 1. For $i \geq 0$, write $Q_{i+1} = Q_i \cup s[Q_i]$. The chain $Q_0 \subseteq Q_1 \subseteq Q_2 \subseteq \dots$ stabilizes after finitely many steps. Indeed, for every rational number $p = a/b \in [0, 1]$ in lowest terms, let us say that b is the *denominator* of p . Then, since the linear pieces of s have all integer coefficients, the denominators of the elements of Q_i are bounded by the maximum denominator d of the elements of Q_0 .

It follows that $\bigcup_i Q_i$ is contained in the finite set of all rational numbers in $[0, 1]$ whose denominator is $\leq d$, and our claim follows.

Display the set of points of $\bigcup_i Q_i$ as $0 = q_0 < q_1 < \dots < q_r = 1$, and let $I_i = [q_{i-1}, q_i]$, for $1 \leq i \leq r$, be the corresponding intervals. On each I_i the function s is of the form $a_i x + b_i$, with $a_i, b_i \in \mathbb{Z}$, and $s[I_i] = I_q \cup I_{q+1} \cup \dots \cup I_{q+t}$, for some $1 \leq q \leq q+t \leq 1$. Define the *Markov graph* of s to be the directed graph G_s whose set of vertices is $\{I_1, \dots, I_r\}$, and there is an edge connecting I_i with I_j iff $s[I_i] \supseteq I_j$. The *Markov matrix* of s is the $r \times r$ matrix E_s whose (i, j) th entry is 0 if $s[I_i] \not\supseteq I_j$ and $|a_i|^{-1}$ otherwise.

A directed graph G is *strongly connected* if every two entries are connected by a path. The *period* of a strongly connected graph is the g.c.d. of the lengths of the paths starting from some given vertex and returning to it; the period does not depend on the choice of the initial vertex. G is *primitive* if it is strongly connected and has period 1. By [33, Theorem 4.5.8] G_s is primitive iff there exists $k \geq 1$ such that E_s^k has all entries > 0 .

Example 3.2. Consider the function s having graph

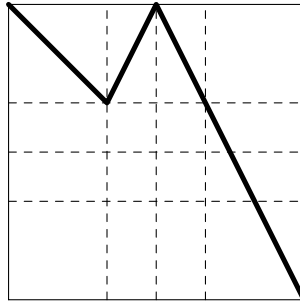


The set of points of nondifferentiability is $Q_0 = \{0, 1/4, 1/3, 1/2, 2/3, 1\}$, which is stable under s . We have

$$E_s = \begin{pmatrix} 1/2 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 1/4 & 1/4 & 1/4 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

and G_s is primitive.

Let now t have graph



We have $Q_0 = \{0, 1/3, 1/2, 1\}$ and $Q_1 = \{0, 1/3, 1/2, 2/3, 1\}$, which is stable. The Markov matrix is

$$E_t = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1/2 \\ 1/2 & 1/2 & 1/2 & 0 \end{pmatrix}$$

and G_t is strongly connected of period 2.

Theorem 3.3. *Let $\sigma : x_0 \mapsto s \in \text{Free}_1(\mathbf{MV})$ be a substitution over 1 variable in Lukasiewicz logic. Then the following are equivalent:*

- (i) σ is generic;
- (ii) σ is ergodic;
- (iii) σ is noninvertible and G_s is strongly connected.

Furthermore, the following are equivalent:

- (iv) σ is mixing;
- (v) σ is exact;
- (vi) σ is noninvertible and G_s is primitive.

If any of (i)–(vi) holds, then the measure $\mu \gg \lambda$ with respect to which s is ergodic is unique and $\mu \ll \lambda$ holds as well. The density function $d\mu/d\lambda \in L_1([0, 1], \lambda)$ has rational values and is constant on each interval I_1, \dots, I_r .

The rest of this section is devoted to the proof of Theorem 3.3. First of all, it is not difficult to show that the only invertible substitutions over 1 variable are the identity and the flip $x_0 \mapsto (x_0 \rightarrow 0)$ [18, Example 2.7]; neither of them is generic. If no path in G_s connects I_i with I_j , then no point of I_i will ever enter the topological interior of I_j , and σ is not generic. This proves (i) \Rightarrow (iii); we already know (ii) \Rightarrow (i) and (v) \Rightarrow (iv).

We prove (iv) \Rightarrow (vi). Assume that σ is mixing (and hence generic), and let t be the period of the strongly connected graph G_s . By [33, §4.5] the vertices of G_s can be partitioned in t equivalence classes $\mathcal{I}_1, \dots, \mathcal{I}_t$ such that the quotient graph (defined by $\mathcal{I}_i \rightarrow \mathcal{I}_j$ iff there exists an edge from some element of \mathcal{I}_i to some element of \mathcal{I}_j) has a cyclic structure $\mathcal{I}_1 \rightarrow \mathcal{I}_2 \rightarrow \dots \rightarrow \mathcal{I}_t \rightarrow \mathcal{I}_1$. By using the fact that s maps the extreme points of the real unit interval to themselves, one sees easily that $t \leq 2$. Assume by contradiction that $t = 2$, and let $A_i = \bigcup \mathcal{I}_i$. Then $s^{-1}[A_1] = A_2$ and $s^{-1}[A_2] = A_1$. This clearly implies that s is not mixing with respect to any measure $\gg \lambda$, which is a contradiction. We conclude that $t = 1$ and (vi) holds.

Lemma 3.4. *Assume (iii) in Theorem 3.3. Then there exists a density function $f \in L_1([0, 1], \lambda)$ such that:*

- f takes strictly positive rational values, and is constant on each interval I_1, \dots, I_r ;
- the measure μ determined by $d\mu = f d\lambda$ is s -invariant.

Proof. Every nonsingular s has finite fibers, and determines the Perron-Frobenius operator [32, Chapter 3]

$$P : L_1([0, 1], \lambda) \rightarrow L_1([0, 1], \lambda)$$

via

$$(Pf)(x) = \sum_{y \in s^{-1}[x]} \frac{f(y)}{|s'(y)|}.$$

If $f \in L_1([0, 1], \lambda)$ has constant value a_i on each I_i , then we identify f with the row vector $(a_1 \cdots a_r)$, and Pf with the row vector $(a_1 \cdots a_r)E_s$. By [30, Theorem 1.3.5], E_s has a real eigenvalue $\alpha > 0$ and a corresponding left eigenvector $(a_1 \cdots a_r)$ (unique up to scalar multiples) such that:

- $\alpha \geq |\beta|$, for every eigenvalue β ;
- $a_i > 0$, for every i .

Write $(b_1 \cdots b_r)^{tr}$ for the column vector defined by $b_i = \lambda(I_i)$, and normalize $(a_1 \cdots a_r)$ by setting $\sum a_i b_i = 1$. Let $f \in L_1([0, 1], \lambda)$ be the density function corresponding to $(a_1 \cdots a_r)$. Since s is surjective, we have

$$\alpha = (a_1 \cdots a_r) E_s \begin{pmatrix} b_1 \\ \vdots \\ b_r \end{pmatrix} = \int_{s^{-1}[0, 1]} (Pf) d\lambda = \int_{[0, 1]} f d\lambda = 1,$$

and f is a fixed point for P (in particular, $(a_1 \cdots a_r) \in \mathbb{Q}^r$). As a consequence, the measure μ determined by

$$\mu(A) = \int_A f d\lambda$$

is s -invariant. Since f is never 0, each of μ and λ is absolutely continuous with respect to the other. \square

Note that if a transformation is ergodic w.r.t. a measure ν , then it does not preserve any other measure $\ll \nu$. In particular, if s is ergodic w.r.t. $\mu \gg \lambda$, then necessarily μ is the measure given by Lemma 3.4. The invariant densities for the functions in Example 3.2 are those corresponding to the vectors $(6/5, 6/5, 9/5, 3/5, 3/5)$ and $(3/4, 3/4, 3/4, 3/2)$, respectively. We present the following well known theorem (see [15, p. 290], [17, Theorem V.2.2], [48, Theorem 12.5]) in a simplified form which is convenient for our needs.

Theorem 3.5. *Let h be a surjective map from $[0, 1]$ to itself, let $0 = q_0 < q_1 < \cdots < q_r = 1$, and let $I_i = [q_{i-1}, q_i]$ for $1 \leq i \leq r$. Assume that:*

- (1) *h is C^2 on $U = [0, 1] \setminus \{q_0, \dots, q_r\}$ and there exist $\alpha, \beta > 0$ such that $\alpha \leq |h'(x)|$ and $|h'(x)|, |h''(x)| \leq \beta$ for every $x \in U$;*
- (2) *there exist $k \geq 1$ and $\gamma > 1$ such that $|(h^k)'(x)| \geq \gamma$ for every x in which the derivative is defined;*
- (3) *every $h[I_i]$ is the union of —necessarily consecutive— intervals in $\{I_1, \dots, I_r\}$, and the resulting Markov graph is primitive.*

Then we have:

- (a) *there exists a unique h -invariant probability measure $\mu \ll \lambda$. Its density $d\mu/d\lambda$ is strictly positive and uniformly bounded away from zero; in particular $\lambda \ll \mu$ as well;*
- (b) *h is exact with respect to μ ;*
- (c) *$\mu(A) = \lim_{n \rightarrow \infty} \lambda(h^{-n}[A])$ for every Borel $A \subseteq [0, 1]$.*

Proof. In order to apply [17, Theorem V.2.2], we only have to check that the conditions on [17, p. 353] are satisfied. This boils down to showing that:

- (4) *there exist $\varepsilon, C > 0$ such that every $h \upharpoonright I_i$ is a $C^{1+\varepsilon}$ diffeomorphism satisfying*

$$\left| \frac{h'(x)}{h'(y)} - 1 \right| \leq C \cdot |h(x) - h(y)|^\varepsilon;$$

(5) there exists $K > 0$ and $\delta > 1$ such that

$$|(h^n)'(x)| \geq K\delta^n,$$

for every $n \geq 1$ and every x for which the derivative is defined.

The bounds in (1) assure that every $h \upharpoonright I_i$ is a $C^{1+\varepsilon}$ diffeomorphism, for $\varepsilon = 1$. We obtain (4) from the intermediate value theorem: $|h'(x)/h'(y) - 1| = |h'(y)|^{-1} \cdot |h'(x) - h'(y)| \leq \alpha^{-1}|h'(x) - h'(y)| \leq \alpha^{-1}\beta|x - y| \leq \alpha^{-2}\beta|h(x) - h(y)|$. If we can take $\alpha > 1$ in (1), then (5) is immediate. Otherwise, let $K = \gamma^{-1}\alpha^{k-1}$ and $\delta = \gamma^{1/k}$. Writing $n = uk + v$ for $0 \leq v < k$ we obtain

$$\begin{aligned} (h^n)'(x) &= (h^v)'(x) \cdot \prod_{j=0}^{u-1} (h^k)'(h^{jk+v}(x)) \\ &\geq \alpha^{k-1}\gamma^u = K\gamma^{u+1} \geq K\delta^n. \end{aligned}$$

□

Assume now (vi) in Theorem 3.3. We claim that $|(s^2)'(x)| \geq 2$ whenever the derivative is defined. For every interval I_i , let $0 \neq a_i \in \mathbb{Z}$ be the value of the derivative of s in I_i . No a_i can be 1, for otherwise the graph G_s would not be strongly connected. Let $I_i \rightarrow I_j$ be an arrow in G_s ; we just need to show that $|a_i a_j| \geq 2$. If this was not the case, then $a_i = a_j = -1$ and $s \upharpoonright I_i = s \upharpoonright I_j = -x + 1$. But then one sees easily that G_s contains the arrow $I_j \rightarrow I_i$ as well, and no other arrow starting from I_i or from I_j . This is impossible since G_s is primitive, and our claim is settled. We can then apply Theorem 3.5, thus obtaining (vi) \Rightarrow (v). To complete the proof of Theorem 3.3, we assume that (iii) holds and prove that s is ergodic w.r.t. the measure μ in Lemma 3.4. If G_s is primitive we are done. Otherwise, by the discussion following the statement of Theorem 3.3, the intervals I_1, \dots, I_r are partitioned in two equivalence classes $\mathcal{I}_1, \mathcal{I}_2$ such that the quotient graph is cyclic. Consider s^2 ; for every edge $a \rightarrow b$ in G_s , let $I_{a,b} = I_a \cap s^{-1}[I_b]$. Then the intervals $I_{a,b}$ are the basic intervals for s^2 . Moreover, $I_{a,b} \rightarrow I_{c,d}$ is an edge in G_{s^2} iff $s^2[I_{a,b}] \supseteq I_{c,d}$ iff $s[I_b] \supseteq I_{c,d}$ iff $s[I_b] \supseteq I_c$ iff $I_b \rightarrow I_c$ is an edge in G_s . This means that G_{s^2} is the disjoint union of two graphs G_1 and G_2 . The vertices of G_1 are the edges of G_s whose starting vertex is in \mathcal{I}_1 , and two such vertices $a \rightarrow b$, $c \rightarrow d$ are connected by an edge in G_1 iff $b \rightarrow c$ is an edge in G_s ; a dual description holds for G_2 .

Lemma 3.6. *Both G_1 and G_2 are primitive.*

Proof. We prove the statement for G_1 . Let $a \rightarrow b$, $c \rightarrow d$ be vertices of G_1 . Since G_s is strongly connected, there is a path in G_s connecting b with c . By looking at the edges of this path alternatively as edges and vertices of G_1 , we obtain a path in G_1 connecting $a \rightarrow b$ with $c \rightarrow d$; hence G_1 is strongly connected. All vertices of a strongly connected graph G have the same period; this means that the period of G can be defined as the g.c.d. of the lengths of the simple closed circuits in G . In our case, the circuits of G_1 correspond bijectively to the circuits of G_s , and this correspondence doubles the lengths. G_s has period 2, and hence G_1 must have period 1. □

The intervals in \mathcal{I}_1 (respectively, \mathcal{I}_2) must be consecutive; indeed, suppose by contradiction $I, J \in \mathcal{I}_1$, and $K \in \mathcal{I}_2$ is between I and J . For some odd $k \geq 1$ we have $s^k[K] \supseteq I \cup J$ and then, since s^k is continuous, $s^k[K] \supseteq K$, which is impossible. Let

$\bigcup \mathcal{I}_1 = [0, q]$ and $\bigcup \mathcal{I}_2 = [q, 1]$. Again by Lemma 3.6 and Theorem 3.5, $s^2 \upharpoonright [0, q]$ and $s^2 \upharpoonright [q, 1]$ are both exact with respect to the appropriate restrictions of the measure μ in Lemma 3.4. We prove that s is ergodic w.r.t. μ by showing that $\lambda(A) > 0$ implies $\lambda(\bigcup \{s^{-n}A : n \geq 0\}) = 1$. Without loss of generality, $B = A \cap [0, q]$ has nonzero measure, and hence so does $s^{-1}B \subseteq [q, 1]$. We have

$$\bigcup_{n \geq 0} s^{-n}A \supseteq \bigcup_{n \geq 0} s^{-n}B = \bigcup_{k \geq 0} s^{-2k}B \cup \bigcup_{k \geq 0} s^{-2k-1}B.$$

The two sets to the right intersect at most in q , and have full measure in $[0, q]$ and in $[q, 1]$, respectively. Therefore, $\bigcup \{s^{-n}A : n \geq 0\}$ has full measure in $[0, 1]$. This completes the proof of Theorem 3.3.

4. FALSUM-FREE PRODUCT LOGIC

In this final section we will discuss the product logic of Example 1.1(4). We will show that the spectrum of its n -generated free algebra is homeomorphic to $\text{Spec Free}_{n-1}(\mathbf{MV})$, with preservation of the measure λ . The substitutions over n variables give rise to continuous piecewise-fractional transformations, and the resulting dynamics is richer than the one in Łukasiewicz logic. In Theorem 4.4 we will see that such substitutions may have attracting fixed points, or may be generic without being ergodic.

Recall from Example 1.1(4) that the falsum-free product logic is defined by the structure $M = ((0, 1], \star, \rightarrow, 1)$, where \star is the product of real numbers and $a \rightarrow b = \min\{1, b/a\}$. In all this section \mathbf{V} will denote the variety generated by M . We assume that the reader is familiar with the basic theory of lattice-ordered abelian groups (ℓ -groups) [22], [7], [5], [16], in particular with the description of the free ℓ -group over n generators $\text{Fl}(n)$ in terms of continuous piecewise-linear homogeneous functions with integer coefficients (plh functions) [2], [3], [4].

The exponential function (say in base e) is an order isomorphism

$$\exp : M \rightarrow ((-\infty, 0], +, \div, 0)$$

between M and the negative cone of \mathbb{R} endowed with the ordinary sum and the dual truncated difference $b \div a = \min\{0, b - a\}$. By [8], \mathbf{V} is exactly the variety of *cancellative Wajsberg hoops*, i.e., algebras $(A, \star, \rightarrow, 1)$ satisfying

$$\begin{aligned} x \rightarrow x &= 1 \\ x \star (x \rightarrow y) &= y \star (y \rightarrow x) \\ x \rightarrow (y \rightarrow z) &= (x \star y) \rightarrow z \\ x &= y \rightarrow (x \star y) \end{aligned}$$

It can be shown [8] that cancellative Wajsberg hoops are categorically equivalent to negative cones of ℓ -groups.

Theorem 4.1. *$\text{Spec Free}_n(\mathbf{V})$ is homeomorphic to $\text{Spec Free}_{n-1}(\mathbf{MV})$.*

Proof. Let $N = (-\infty, 0]^n$ be the negative orthant of \mathbb{R}^n , and let P be the polyhedral cone spanned positively by $\{b_1 e_1 + b_2 e_2 + \dots + b_{n-1} e_{n-1} + e_n : b_1, \dots, b_{n-1} \in \{0, 1\}\}$, where $(e_1 \dots e_n)$ is the standard basis of \mathbb{R}^n . Let I and J be the principal ideals of $\text{Fl}(n)$ whose elements are all plh functions which are 0 in N and in P , respectively. Form the quotient ℓ -groups $\text{Fl}(n)/I$ and $\text{Fl}(n)/J$, and let $(\text{Fl}(n)/I)^-$, $(\text{Fl}(n)/J)^-$

be their negative cones. $(\text{Fl}(n)/I)^-$ can be identified with the set of all plh functions from N to $(-\infty, 0]$, and analogously for $(\text{Fl}(n)/J)^-$. We have:

- the categorical equivalence between cancellative Wajsberg hoops and negative cones of ℓ -groups associates $\text{Free}_n(\mathbf{V})$ to $(\text{Fl}(n)/I)^-$; this is proved, in dual form, in [14]. The spectrum is preserved both by the equivalence and by passing from an ℓ -group to its negative cone (because every ℓ -group homomorphism is determined by its behaviour on the negative cone). Therefore $\text{Spec Free}_n(\mathbf{V})$ is homeomorphic to $\text{Spec}(\text{Fl}(n)/I)$ (see [5, Chapter 10] for the spectra of ℓ -groups);
- by [46, p. 195], $\text{Spec}(\text{Fl}(n)/J)$ is homeomorphic to $\text{Spec Free}_{n-1}(\mathbf{MV})$.

We will establish our claim by showing that $\text{Fl}(n)/I$ and $\text{Fl}(n)/J$ are isomorphic. By [4] this can be done by triangulating N and P into combinatorially isomorphic complexes of unimodular cones (a polyhedral cone is *unimodular* if it is positively spanned by a \mathbb{Z} -basis of \mathbb{Z}^n).

Let R be obtained from the $n \times n$ identity matrix by permuting the first $n - 1$ rows. Let C_R be the unimodular cone positively spanned by the columns of

$$R = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 0 & 1 & \cdots & 1 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{pmatrix}$$

The collection of all faces of all the $(n-1)!$ C_R 's is a complex that triangulates P [43, Lemma 2.1]. Analogously, let D_R be the cone spanned by the columns of

$$R = \begin{pmatrix} 0 & -1 & -1 & \cdots & -1 & -1 \\ 0 & 0 & -1 & \cdots & -1 & -1 \\ 0 & 0 & 0 & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -1 \\ -1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

Again D_R is unimodular and the complex of all D_R 's triangulates N . This latter fact is easily seen by observing that D_R contains exactly those vectors of N whose first $n - 1$ coordinates $\alpha_1, \dots, \alpha_{n-1}$ satisfy $\alpha_{\rho^{-1}1} \leq \alpha_{\rho^{-1}2} \leq \cdots \leq \alpha_{\rho^{-1}(n-1)}$, where ρ is the permutation that originated R . The unimodular complexes thus obtained are clearly combinatorially isomorphic, and this concludes the proof. \square

It is convenient to realize $\text{Free}_n(\mathbf{V})$ as $(\text{Fl}(n)/I)^-$. Let Δ^{n-1} be the $(n - 1)$ -dimensional simplex $\{\sum \alpha_i e_i : \sum \alpha_i = -1\}$. The mapping ρ that associates to a point $u \in \Delta^{n-1}$ the ideal $\rho(u)$ of all plh functions which are 0 in u is a homeomorphic embedding of Δ^{n-1} into $\text{Spec Free}_n(\mathbf{V})$, whose range is precisely the set of maximal ideals. Let $\sigma : \text{Free}_n(\mathbf{V}) \rightarrow \text{Free}_n(\mathbf{V})$ be a substitution. Then we have a commuting

diagram

$$\begin{array}{ccc} \Delta^{n-1} & \xrightarrow{\tilde{s}} & \Delta^{n-1} \\ \rho \downarrow & & \downarrow \rho \\ \text{Spec Free}_n(\mathbf{V}) & \xrightarrow{S} & \text{Spec Free}_n(\mathbf{V}) \end{array}$$

where \tilde{s} is the piecewise-fractional transformation defined as follows: if $\sigma(x_i) = s_i(x_0, \dots, x_{n-1})$ (s a polynomial in the language $(+, \div, 0)$) and $u = (\alpha_0, \dots, \alpha_{n-1})$, then $\tilde{s}(u) = -(\sum s_i(u))^{-1}(s_0(u), \dots, s_{n-1}(u))$.

On $\text{Spec Free}_n(\mathbf{V})$ we have, a priori, two reasonable measures: our standard λ obtained by pushing forward via π the Lebesgue measure on $[0, 1]^n$, and an unnamed measure obtained by pushing forward via ρ the Lebesgue measure on Δ^{n-1} .

Theorem 4.2. *The unnamed measure coincides with λ .*

Proof. It suffices to prove the statement for the basic closed sets $F_t = \text{Spec Free}_n(\mathbf{V}) \setminus O_t$. Let us write t_1 for the polynomial t written in the language $(\star, \rightarrow, 1)$, and t_2 for the same polynomial written in the language $(+, \div, 0)$. Then $\lambda(F_t)$ is the n -dimensional Lebesgue measure of $Z(t_1) \subseteq [0, 1]^n$, while the unnamed measure of F_t is the $(n-1)$ -dimensional Lebesgue measure, say r , of the section $Z(t_2) \cap \Delta^{n-1}$ of the polyhedral cone $Z(t_2)$. The componentwise exponential function gives a diffeomorphism

$$\exp : (-\infty, 0]^n \rightarrow (0, 1]^n,$$

and $\exp^{-1}[Z(t_1)] = Z(t_2)$. We have therefore

$$\lambda(Z(t_1)) = \int_{Z(t_2)} |J| dx_0 \dots dx_{n-1},$$

where the Jacobian J of the diffeomorphism has determinant $\exp(x_0 + \dots + x_{n-1})$ in the point (x_0, \dots, x_{n-1}) . By parametrizing $Z(t_2)$ along a ray, the Riemann integral to the right reduces to the improper integral

$$\int_{-\infty}^0 -x r \exp(x) dx,$$

which has value r ; this concludes the proof. \square

Theorem 4.2 makes it possible to formulate the analogue of Theorem 3.1: the proof carries over with straightforward modifications.

Theorem 4.3. *If any of the mappings \tilde{s} and S is nonsingular, then so is the other. If this happens, then the systems $(\Delta^{n-1}, \tilde{s}, \text{Lebesgue measure})$ and $(\text{Spec Free}_n(\mathbf{V}), S, \lambda)$ are isomorphic under ρ .*

We conclude our paper with an example. Choose integers $a, b \geq 1$, and let σ be the substitution over two variables defined by

$$\begin{aligned} x_0 &\mapsto [((x_0 \rightarrow x_1) \rightarrow x_1) \rightarrow x_0 \star (x_0 \rightarrow x_1)]^a \\ x_1 &\mapsto [(x_0 \rightarrow x_1) \rightarrow x_1]^b \end{aligned}$$

(t^a means $t \star t \star \dots \star t$ a times). Let $\bar{s} : [0, 1]^2 \rightarrow [0, 1]^2$ be as in Section 2, and let $\tilde{s} : \Delta^1 \rightarrow \Delta^1$ be as in this section. Note that

$$\begin{aligned} s_0 &= \sigma(x_0) = a[(x_0 + (x_1 \div x_0)) \div (x_1 \div (x_1 \div x_0))] \\ &= a[(x_0 \wedge x_1) \div (x_0 \vee x_1)] = a[(x_0 \wedge x_1) - (x_0 \vee x_1)] \\ s_1 &= \sigma(x_1) = b[x_1 \div (x_1 \div x_0)] = b[x_0 \vee x_1] \end{aligned}$$

as plh functions from $(-\infty, 0]^2$ to $(-\infty, 0]$. It is convenient to identify Δ^1 with $[0, 1]$ via $x \mapsto (-1 + x, -x)$. Under this identification \tilde{s} is a transformation on $[0, 1]$ which depends only on the ratio $q = a/b > 0$. Indeed, a straightforward computation shows that

$$\tilde{s}(x) = \begin{cases} x \cdot ((1 - 2q)x + q)^{-1} & \text{if } 0 \leq x \leq 1/2; \\ (1 - x) \cdot ((1 - 2q)(1 - x) + q)^{-1} & \text{if } 1/2 < x \leq 1. \end{cases}$$

The graph of \tilde{s} is tent-like; since $\tilde{s}(x) = \tilde{s}(1 - x)$, it is symmetric with respect to the line $x = 1/2$. For $q = 1/2$ the slopes of the tent are straight lines, for $1/2 < q$ they are convex functions, and for $q < 1/2$ are concave.

Theorem 4.4. *Let $a, b, q, \sigma, \bar{s}, \tilde{s}$ be as above.*

- (1) *If $q > 1$, then λ -all points of $[0, 1]$ are attracted to 0 under iteration of \tilde{s} .*
- (2) *If $q < 1$, then \tilde{s} is exact with respect to a uniquely determined measure $\ll \lambda$.*
- (3) *If $q = 1$, then*
 - (3.1) *σ is generic;*
 - (3.2) *no point of $(0, 1]^2$ has a dense orbit under \bar{s} ;*
 - (3.3) *\tilde{s} is not ergodic with respect to any probability measure $\gg \lambda$.*

Proof. Let $f = \tilde{s} \upharpoonright [0, 1/2]$, and note that $\tilde{s}(x) = f(\min(x, 1 - x))$. We have $f'(x) = q((1 - 2q)x + q)^{-2}$; in particular $|f'(0)| = 1/q$, and (1) is immediate. We prove (2) by applying Theorem 3.5; the only condition that needs checking is the second. If $1/2 \leq q$, then $f'(x) \geq f'(0) > 1$ is easily shown. Otherwise, if $q < 1/2$, then $(1 - 2q)x + q$ is positive strictly increasing on $[0, 1/2]$ and hence $f'(x)$ is positive strictly decreasing on the same interval. Let $d = f^{-1}(1/2) = q/(2q + 1)$. We have $\tilde{s}^2(x) = f^2(x)$ on $[0, d]$ and $\tilde{s}^2(x) = f(1 - f(x))$ on $[d, 1/2]$. By the chain rule, $|(\tilde{s}^2)'| \geq f'(1/2) \cdot f'(d) = (2q + 1)^2 > 1$ on $[0, d]$. We claim that the same bound holds on $[d, 1/2]$ (and hence on all $[0, 1]$). By direct computation

$$\left| \frac{d}{dx} f(1 - f(x)) \right| = \frac{1}{((2q - 1)x - (q - 1))^2},$$

and the denominator describes an upward parabola having vertex in a point $> 1/2$. Hence $|(\tilde{s}^2)'|$ is bounded from below in the interval $[d, 1/2]$ by the value $(2q + 1)^2$ that the displayed expression assumes in d , as claimed.

We prove (3) starting from (3.2). Let $p = (\alpha, \beta) \in (0, 1]^2$ be on the hyperbola $x_0 x_1 = c$. Then $\bar{s}(p) = ((\alpha \wedge \beta)/(\alpha \vee \beta), \alpha \vee \beta)$ is on $x_0 x_1 = d$, where $d = \alpha \wedge \beta \geq \alpha \beta = c$. It follows that the orbit of p is all above the hyperbola $x_0 x_1 = c$, and hence it cannot be dense in $(0, 1]^2$.

The proof of (3.1) requires a few basic facts about continued fractions: see [26] or [51]. Every real number α has a unique expansion into a continued fraction

$$\alpha = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \dots}} = [a_0, a_1, a_2, \dots];$$

the expansion is finite iff α is rational. The *Gauss map* $g : [0, 1] \rightarrow [0, 1]$ is defined by $g(0) = 0$ and $g(\alpha) = 1/\alpha - [1/\alpha]$ if $\alpha \neq 0$. If α has the above expansion, then $g(\alpha)$ has the expansion $[0, a_2, a_3, \dots]$. Suppose that $\alpha \in \{0\} \cup (1/2, 1]$; this means that either $\alpha = [a_0]$ (i.e., α equals 0 or 1), or $a_0 = 0$ and $a_1 = 1$. Then $\tilde{s}(\alpha) = g(\alpha)$ (by definition if $\alpha \in \{0, 1\}$, and because $\tilde{s}(\alpha) = (1 - \alpha)\alpha^{-1} = \alpha^{-1} - 1 = [a_1, a_2, a_3, \dots] - 1 = [0, a_2, a_3, \dots] = g(\alpha)$ if $\alpha \in (1/2, 1)$). Suppose on the other hand $\alpha = [0, a_1, a_2, \dots] \in (0, 1/2]$, so that $a_1 \geq 2$. Then $\tilde{s}(\alpha) = [0, a_1 - 1, a_2, \dots]$; indeed, $(\tilde{s}(\alpha))^{-1} = \alpha^{-1} - 1 = [a_1, a_2, a_3, \dots] - 1 = [a_1 - 1, a_2, a_3, \dots] = [0, a_1 - 1, a_2, \dots]^{-1}$.

We see therefore that \tilde{s} acts like a slow continued fraction algorithm. Starting from α and applying \tilde{s} we touch every point which is touched by g , but at a slower rate: when we reach, say, $\beta = [0, b_1, b_2, \dots]$ with b_1 a large number, we must move to all points $[0, b_1 - c, b_2, \dots]$ for $1 \leq c < b_1$, before reaching $g(\beta)$. In particular, for every α , the \tilde{s} -orbit of α contains the g -orbit of α . Now, the Gauss map is ergodic with respect to the Gauss measure γ , which has the same nullsets as the Lebesgue measure λ [6, Section 4]. This implies that λ -all α 's have a dense g -orbit, and therefore a dense \tilde{s} -orbit as well; by Theorem 4.3 σ is generic.

A proof of (3.3) can be adapted from the proof in [32, Remark 6.2.1] for a similar transformation. However, this requires a rather delicate analysis of the convergence of the sequence of densities $\{P^n 1\}$, where P is the Perron-Frobenius operator induced by \tilde{s} . We give a proof based on continued fractions. Assume by contradiction that \tilde{s} is ergodic w.r.t. the probability measure $\mu \gg \lambda$. Let $r = \mu((1/2, 1]) > 0$. By the Birkhoff Ergodic Theorem [53, §1.6], for μ -all points α we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} \text{card}\{i : 0 \leq i < n \text{ \& } \tilde{s}^i(\alpha) \in [0, 1/2]\} = 1 - r. \quad (*)$$

By [6, p. 45], for λ -all points $\alpha = [0, a_1, a_2, \dots]$ we have

$$\lim_{k \rightarrow \infty} \frac{k}{a_1 + \dots + a_k} = 0. \quad (**)$$

Pick an α that satisfies both (*) and (**), and write $n_k = a_1 + \dots + a_k$; note that the expansion of α must be infinite. Passing to a subsequence in (*) we get

$$\lim_{k \rightarrow \infty} \frac{1}{n_k} \text{card}\{i : 0 \leq i < n_k \text{ \& } \tilde{s}^i(\alpha) \in [0, 1/2]\} = 1 - r.$$

Since $[0, b_1, b_2, \dots] \in [0, 1/2]$ iff $b_1 \geq 2$, we have

$$\text{card}\{i : 0 \leq i < n_k \text{ \& } \tilde{s}^i(\alpha) \in [0, 1/2]\} = \sum_{i=1}^k (a_i - 1) = n_k - k.$$

We conclude that

$$1 - r = \lim_{k \rightarrow \infty} (1 - k/n_k) = 1.$$

This is a contradiction since $r > 0$. □

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